Avalanche dynamics in model two-dimensional grain piles

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We report particle dynamical simulations of a two-dimensional grain pile in a box with the base being slowly tilted from horizontal to Θ_{aval} angle at which the pile undergoes a large layer sliding event. When dissipation between the grains is negligible, the distribution $D(s)$ of displacements s of the surface grains shows decays that are consistent with $s^{-\tau}$ with $\tau \approx 2$ for $0<\Theta<\Theta_{\mathrm{aval}}$. At time $t>t_{\mathrm{aval}}$ at Θ_{aval} we find a crossover in τ to $\tau \approx 3/2$. Dissipation appears to play a key role in the system dynamics only when $\Theta<\Theta_{\mathrm{aval}}$. We find that the time for the onset of avalanches, $t_{\mathrm{aval}} \geqslant t_W$, where t_W is the time when the surface roughness of the sliding pile is minimized. [S1063-651X(97)04511-X]

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The dynamical behavior of dry granular materials such as sand, gravel, salt, and the like have fascinated scientists and engineers for more than two centuries [1,2]. This fascination has in turn resulted in significant progress in the understanding of the packing of granular materials and in describing flowing granular systems [2,3]. However, much remains to be learned about the detailed dynamical processes that are responsible for the onset of instabilities in these materials. In this paper we report on the dynamics of a granular system near the onset of instabilities (i.e., near a ''critical point'') and the associated crossover behavior exhibited by the system. One would expect that the dynamical behavior exhibited in the vicinity of a critical point will be independent of the details of the interactions between the grains, and that only the minimal essential properties will contribute to its critical behavior. The key results in this work confirm this expectation. Although the dynamics near the instability depend quite strongly on the details of the interactions and the dissipations used (e.g., restitution, see models A and B below), we find that the fluctuations in the system dominate these details at the point of instability.

For the sake of simplicity, we consider two-dimensional (2D) models consisting of granular disks. We study the dynamics in two different model systems at minimal and at significant magnitudes of the restitution coefficient. We start with a triangular lattice of granular disks that have been placed in a horizontally oriented rectangular box. The box is tilted infinitesimally slowly (i.e., ''adiabatically'') until a characteristic threshold angle, Θ_{aval}, is reached. We find that in general, under adiabatic tilting conditions, the magnitude of Θ_{aval} and the details of the dynamical process of onset of sliding of the surface layer grains at tilts below Θ_{aval} depend upon the interactions and the strength of dissipation. We have ignored the presence of static friction and hence of rotation of the disks in this analysis. One would expect that the rotation of the grains will affect the dynamical behavior of the surface of the grain pile at subcritical tilt angles. The effects of rotation will be addressed in a separate study.

However, our calculations reveal that in the vicinity of Θ_{aval}, dissipation has marginal influence on the system dynamics. We find that at Θ_{aval}, the upper layers of the pile become unstable, which, in turn, leads to a large layer sliding event or an avalanche. A ''back of the envelope'' calculation of the importance of rotation of the grains at Θ_{aval} reveals that $\omega \sim 10^3$/sec for rotational kinetic energies to be comparable to translational kinetic energies [4] at the start of layer sliding for millimeter sized grains of mass 10^{-1} g at terrestrial gravity. Thus, we contend that the avalanche dynamics at criticality is not significantly affected by ignoring grain rotation. As we shall see, the impressive agreement between our results, which are obtained without accounting for grain rotation, versus those from the available experiments supports our contention. The avalanche occurs after a time t_{aval} has elapsed since the system has been adiabatically raised to Θ_{aval}. The focus of this article is to understand the processes that determine Θ_{aval} and t_{aval}.

Our studies on model 2D grain piles lead to results that compare favorably with available experimental data on 3D granular systems at the verge of a layer sliding event [5]. In closing, we comment briefly on the correspondence between the data in 2D and 3D systems and make specific predictions concerning t_{aval} that may be experimentally tested.

System Preparation. We start with a triangular lattice in a rectangular box with 4 layers and 298, 300, 298, and 300 grains in the layers when counted from the surface. In addition, we impose a corrugation potential upon which the fourth layer is registered at zero tilt. This potential adequately mimics the effects of the immobile deeper layers in the pile [4]. The system is subjected to a gravitational field. The box is first kept at zero tilt (i.e., with its base parallel to the horizon) and is progressively tilted in steps of 1°. At each tilt, the system is allowed to evolve in time until dissipation arrests all granular motion. This process is repeated until Θ_{aval} is found, at which a major layer sliding event occurs. The time evolution of the surface region of the system is affected by static friction at short times. Static friction, and hence rotational effects, weakly affect the surface dynamics near the onset of avalanches when the translational kinetic energies of grains dominate the contributions to the rotational kinetic energies [4]. Thus, for the sake of simplicity, we ignore the effects of grain rotation in this study. As we shall see, our results on the distances traveled by the grains at finite tilts compare favorably with experiments and hence support our assumption. We discuss below the details of the interactions and the dissipations in the two sets of models that we study.

(A) Model with minimal dissipation. In our first study we consider disks that repel via a hard-core-like potential $V(|\vec{r}_{ij}|)$ ($|\vec{r}_{ij}| \equiv |\vec{r}_i - \vec{r}_j|$, $\vec{r}_i \equiv \{x_i, y_i\}$), and are subjected to the external gravitational field described by $V(r_i)$. The system is therefore described by a total potential $V = V(|\vec{r}_{ij}|) + V(r_i)$ [4], where

$$V(|\vec{r}_{ij}|) = \frac{\epsilon}{1-6/\epsilon}\left\{\frac{6}{\alpha}\exp\left[\alpha\left(1-\frac{|\vec{r}_{ij}|}{\sigma}\right)\right]-\left(\frac{\sigma}{|\vec{r}_{ij}|}\right)^6\right\}+\epsilon,$$

$$V(r_i) = mgy_i. \qquad (1)$$

We set $\epsilon = 2.9697\times10^{-4}$ kg m/s^2, $\alpha = 55$ (strong repulsive core), $m = 4.4\times10^{-4}$ kg, and $\sigma = 2r$, where $r = 10^{-3}$ m is the radius of each uncompressed grain. The second equation above describes the gravitational potential energy of each grain, y_i being the height of the grain i with respect to some chosen zero of the potential energy.

We include energy dissipation in our analysis of model A [4]. The dissipation enters through the dynamics of those grains that collide with the walls of the box and lose a fraction η of their kinetic energies [4]. In this form of dissipation, the energy is lost slowly in time when compared to the typical time scale of granular motion. The slow energy loss prevents the system from melting and eventually freezes it in a disordered state. We denote the dissipation forces by $\vec{\Lambda}(\eta, \vec{r}_i)$ in the equations of motion for the grains in model A. Therefore,

$$m\ddot{\vec{r}}_i = -\left\{\vec{\nabla}_i\left[\sum_j V(|\vec{r}_{ij}|)+V(r_i)\right]+\vec{\Lambda}(\eta,\vec{r}_i)\right\}. \qquad (2)$$

We ignore the effects of restitution in model A. We shall return to the role of restitution in avalanche dynamics when we consider model B below, which will possess a more realistic form of $V(|\vec{r}_{ij}|)$.

We integrate the system of equations in Eq. (3) using a "velocity Verlet algorithm" [6]. Due to the short interaction range of the potential and the significant influence of gravity on the grains, we find that a very small integration time step, $\Delta t \approx 10^{-7}$ sec, is necessary for dynamical studies. The system is time integrated typically up to $t_{\max} \approx 1$ sec at each tilt. We have studied the process of onset of avalanches for two cases of Eq. (3). One with $\eta = 0.95$ and the other with $\eta = 0.50$. Our results are summarized below.

We note that at finite tilts, the surface of a grain pile resembles a rugged terrain, i.e., a highly aperiodic surface. Let s denote the distance traveled by a grain at some tilt Θ and let $D(s)$ describe the number of grains that travel s at that tilt. We expect that at large s, $D(s)$ will decay in s. The profile of the surface suggests that we should search for an algebraic decay in $D(s)$, say of the form

$$D(s) \sim s^{-\tau}, \quad \tau > 0, \quad s \gg r. \qquad (3)$$

From our simulations, we find that at $6 < \Theta < \Theta_{\rm aval}$, $\tau \approx 2$. It becomes progressively difficult to accurately estimate τ from the simulations for $\Theta < 6°$.

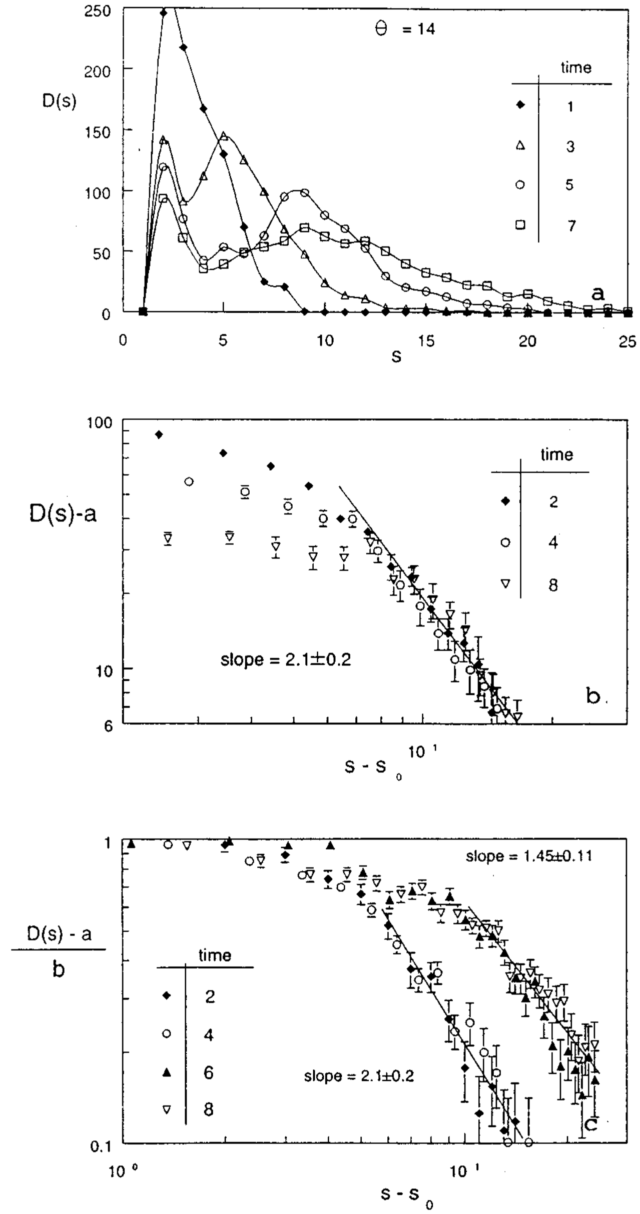

FIG. 1. (a) Behavior of $D(s)$ vs s (in units of $1.6r$) at different times (in units of $\tau_0 = 0.0125$ sec.) at $\Theta = 14°$ for model A (see text); (b) we assume $D(s) = a + b/(s-s_0)^\tau$ for large s. Here, for a given Θ, a is a constant and b depends on time. The data show $D(s)$ decays as $(s-s_0)^{-\tau}$ at $\Theta = 14°$; (c) $D(s)$ at $\Theta_{\rm aval} = 17°$ at $t < t_{\rm aval}$ and at $t \geq t_{\rm aval}$.

The behavior of $D(s)$ versus s at $\Theta = 14°$ is shown in Fig. 1(a). In this simulation we chose $\eta = 0.95$. Figure 1(a) suggests that at short enough times after the pile has been tilted, the surface layer tends to establish registry with the layer below by undergoing collective motion of the grains over small length scales. This explains the large peak in $D(s)$ at s_0 at short times. The magnitude of s_0 increases somewhat in time and is a function of Θ. It may be noted that very similar observations were made by Bretz *et al.* [5].

As $\Theta \rightarrow \Theta_{\rm aval}$, the system grains at the surface undergo larger length scale motion to release the excess potential energy they have gained during the incremental tilt. This in turn results in less motion at small length scales and significant motion at larger length scales at late times [Figs. 1(a) and 1(b)]. $D(s)$ versus s is shown in a log-log plot in Fig. 1(b) and yields a slope of

$$\tau = 2.1 \pm 0.2, \quad \Theta \leqslant \Theta_{\mathrm{aval}}. \tag{4}$$

After the tilt is adiabatically raised by several degrees, it turns out that we reach $\Theta_{\mathrm{aval}} = 17°$. We found that at short times τ continues to be at the same value as before. However, upon waiting sufficiently long, a layer sliding event commences at some time t_{aval}. For $t > t_{\mathrm{aval}}$, τ suffers a crossover to a slope

$$\tau = 1.45 \pm 0.11, \quad \Theta = \Theta_{\mathrm{aval}}, \quad t > t_{\mathrm{aval}}. \tag{5}$$

The properties of $D(s)$ described above were found in all the 2D systems we studied, which can be described under model A.

The range over which we found the scaling behavior described above is small. To extend the range in s without considering larger systems we probed a model A system with $\eta = 0.5$. The smaller η in this study allowed for more motion of the surface grains and hence for an improved range across which the scaling behavior of τ could be seen. In this study we found an exponent $\tau = 1.98 \pm 0.06$ at $t < t_{\mathrm{aval}}$ and a $\tau = 1.48 \pm 0.08$ immediately after the commencement of the avalanche. Our simulations suggest that for sufficiently high Θ (typically, $\Theta > 6°$ or so) in systems with small dissipation (such as energy dissipation that occurs slowly through the walls), $\tau \approx 2$ for $\Theta < \Theta_{\mathrm{aval}}$ and that $\tau \approx 1.5$ when $\Theta = \Theta_{\mathrm{aval}}$.

Since the avalanche occurs a characteristic time after the system is raised to Θ_{aval}, we hypothesized that the roughness W of the system is minimized at some t_W and that $t_W < t_{\mathrm{aval}}$, i.e., the time in which t_{aval} commences at Θ_{aval}. We characterized the dynamically evolving roughness of the system via

$$W(t) \equiv \sum_{i=1}^{N'} \sqrt{\langle [h(\vec{r}_i,t) - \langle h \rangle]^2 \rangle}, \tag{6}$$

with

$$\langle h \rangle \equiv \frac{1}{\Delta} \int_0^{\Delta} dt' \frac{1}{N'} \sum_{i=1}^{N'} h(\vec{r}r_i, t'), \tag{7}$$

where $h(\vec{r}_i,t)$ is the height of a grain at location $\vec{r}_i$ and at time t with respect to the base of the box and N' ($\sim$900) is the number of grains that lie in the ''bulk'' region of the grain pile, i.e., which are unaffected by the boundaries. $\langle h \rangle$ above is constructed by averaging over the grain positions in 10 distinct ''snapshots'' of the system taken across uniformly spaced time intervals over a time window Δ.

We expect that

$$dW(t)/dt = 0, \quad d^2W(t)/dt^2 > 0, \tag{8}$$

at some time t_W at Θ_{aval}, i.e., at time t_W, the surface possesses the minimum roughness. The time t_{aval} at which the avalanche commences should then equal or should be slightly larger than t_W. Our results are consistent with this expectation. We find that $W(t)$ reaches a minimum between $t \approx 0.05$ sec and 0.075 sec (see Fig. 2) at $\Theta = \Theta_{\mathrm{aval}} = 17°$. From Fig. 2, which represents data accumulated over a period of time (rather than instantaneous data), we conclude that the avalanche commences after 0.05 sec and before 0.075 sec. We shall show below that similar conclusions are reached with different potentials and dissipation mechanisms.

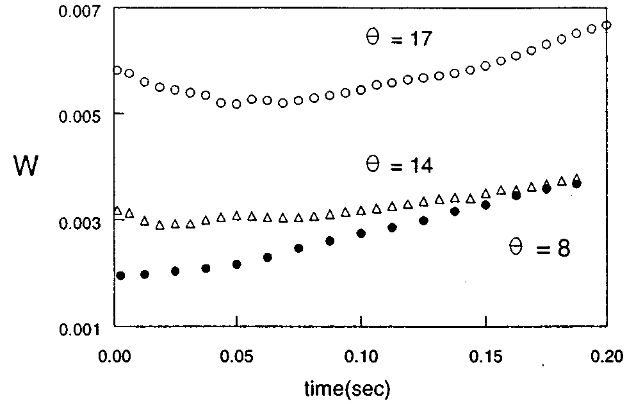

FIG. 2. Plot of roughness $W(t)$ vs t at $\Theta = 8$, 14 and and at $\Theta_{\mathrm{aval}} = 17°$. The open circles indicate the development of a minima at t_{aval}.

(B) Modeling the role of restitution. A standard way to model a granular system is by using the Hertzian contact law in which two particles i and j interact only if they are in contact in the manner described below [7]. If $|\vec{r}_{ij}|$ is the distance between the centers of nearest neighbor particles, then the force on particle i exerted by particle j is given as $\vec{F}_{ij} = F_{n,ij}\hat{n} + F_{s,ij}\hat{s}$. Here $\hat{n}$ is the component of the force along the line joining the centers of the two nearest neighbor grains and $\hat{s}$ is orthogonal to $\hat{n}$. We neglect the shear component of the force, i.e., $F_{s,ij}$. The presence of this term introduces static fiction. $F_{n,ij}$ is responsible for velocity dependent dissipation and is included in our study. Thus, we consider

$$F_{n,i}\hat{n} = \left[\sum_{j(j>i)} k_n(\sigma - |\vec{r}_{ij}|)^{3/2}\hat{n} - \sum_j \vec{f}_{ij,\mathrm{diss}} \right] - m\vec{g} - \vec{\Lambda}(\eta, \vec{r}_i), \tag{9}$$

$k_n = 0$ if $\sigma < |\vec{r}_{ij}|$, where the first term on the right hand side is the Hertzian repulsion due to the deformation of the grains [7]. The constant k_n is set to 10^6 N/m$^{3/2}$. The second term on the right in Eq. (10) is the velocity dependent friction. We shall assume a simple form for $\vec{f}_{ij,\mathrm{diss}}$ given as

$$|\vec{f}_{ij,\mathrm{diss}}| = \frac{\gamma_n m}{2} |([\dot{\vec{r}}_i - \dot{\vec{r}}_j] \cdot \hat{n})|, \tag{10}$$

where m is the mass of each grain, and γ_n is a damping constant. We have studied model B for $\gamma_n = 0$ and find that our results for τ are identical to those from model A. γ_n can be approximately related to the coefficient of restitution ϵ $(0 \leqslant \epsilon \leqslant 1)$ via the relation [8]

$$\gamma_n \approx -\ln \epsilon / \sqrt{\pi^2 + \ln^2 \epsilon}. \tag{11}$$

In our study we chose an effective coefficient of restitution of 0.75.

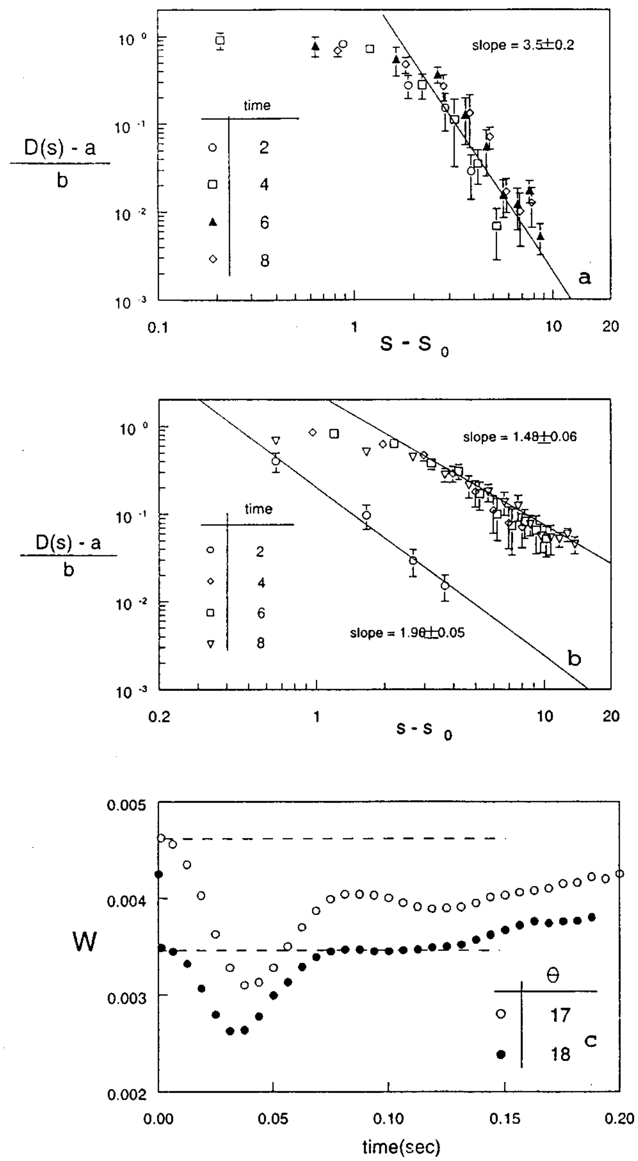

FIG. 3. (a) Plot of $D(s)$ vs s (in units of $2.0r$) at $\Theta=17°$ in model B; (b) $D(s)$ at $\Theta_{\text{aval}}=18°$ using model B. Note that dissipation plays no significant role at Θ_{aval}; (c) $W(t)$ versus t at $\Theta=17°$ and at $\Theta_{\text{aval}}=18$ deg.

We find that for model B at $\Theta<\Theta_{\text{aval}}$, the dynamics of the system is very different compared to that at $\Theta_{\text{aval}}-\Theta\rightarrow 0$. The value of τ at a high but subcritical angle of $\Theta=17°$, for example, is 3.5 ± 0.2, which far exceeds $\tau\approx 2$ found at a comparable subcritical angle in model A [see Fig. 3(a)]. Our calculations show $\tau=1.96\pm 0.05$ at $t<t_{\text{aval}}$ at $\Theta_{\text{aval}}=18$ deg in model B. We also find that τ crosses over to $\tau=1.48\pm 0.06$ at $t\approx t_{\text{aval}}$ at $\Theta_{\text{aval}}=18$ deg [Fig. 3(b)].

It is instructive to do some simple analysis to understand the crossover behavior in τ as an avalanche commences [9]. Let us consider a staircase of blocks that can topple upon the slightest perturbation. Let L be the linear dimension of the system and $D(s,L)$ be the probability that adding a block at random causes s topplings (distance unit$=1$) before a new stable configuration is reached. Also, on average, one block leaves a pile as a result of adding a block. In this critical state, adding a grain results in structural rearrangements at least of the order of system size (layer flow). Let us further assume that in this critical state, the average number of topplings undergone by a labeled grain before it leaves the system, a global quantity, equals the average number of topplings caused per added grain, a local quantity. Let this number be s_m. In the critical state, it is reasonable to assume that $s_m\sim L$. Then the first moment of $D(s,L)$,

$$M_1\equiv\sum_{s=0}^{L} sD(s,L)\geqslant kL, \quad k>0. \tag{12}$$

If we let $L\rightarrow\infty$, then $M_1\rightarrow\infty$. If we suppose that $D(s,\infty)$ in this disordered system has an algebraic tail, i.e.,

$$D(s,\infty)\rightarrow 1/s^{1+a}, \tag{13}$$

for large s, then

$$a<1, \quad M_1\rightarrow\infty. \tag{14}$$

Indeed, mean field theory calculations show that $\tau=3/2$ [10], which is exactly what our calculations on the model system give for τ when layer sliding begins. When $M_1<\infty$, $\tau\geqslant 2.0$. This leads to

$$D(s,L)\sim 1/s^{1+a}, \quad a>1 \tag{15}$$

for large s, consistent with our findings and those in the experiments in [5].

Given that the gravitational field plays a dominant role in dictating the system dynamics and that the grains interact only upon contact it is not surprising that our results for 2D systems are consistent with the available experimental data on monodisperse 3D grain piles. Finally, based upon our calculations, we predict that the time of onset of an avalanche t_{aval} in adiabatically tilted grain piles is bounded from below by the time at which the surface of the grainpile is at its smoothest.

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[1] C.-A. de Coulomb, Acad. R. Sci. Mem. Math. Phys. **7**, 343 (1773).
[2] See, for instance, *Grains and Powders 97*, edited by R. P. Behringer and J. T. Jenkins (A. A. Balkema, Rotterdam, 1997).
[3] O. R. Walton, Mech. Mater. **16**, 239 (1993); O. R. Walton and R. L. Braun, J. Rheol. **30**, 949 (1986).
[4] S. Sen, N. Mousseau, and G. Overney, Phys. Rev. E **49**, 4712 (1994); S. Sen and S. Pal (unpublished).
[5] G. A. Held *et al.*, Phys. Rev. Lett. **65**, 1120 (1990); J. Rosendahl *et al.*, *ibid.* **73**, 537 (1994); M. Bretz *et al.*, *ibid.* **69**,

2431 (1992); V. Frette *et al.*, Nature (London) **379**, 49 (1996).

[6] J. M. Haile, *Molecular Dynamics Simulations* (Wiley, New York, 1993).

[7] J. D. Goddard, Proc. R. Soc. London, Ser. A **430**, 105 (1990); R. S. Sinkovits and S. Sen, Phys. Rev. Lett. **74**, 2686 (1995); S. Sen and R. S. Sinkovits, Phys. Rev. E **54**, 6857 (1996).

[8] G. H. Ristow, Int. J. Mod. Phys. C **3**, 1281 (1992).

[9] D. Dhar, Physica A **186**, 82 (1992); S. Sen and S. Pal, *ibid.* **233**, 77 (1996).

[10] S. Zapperi, K. B. Lauritsen, and H. E. Stanley, Phys. Rev. Lett. **75**, 4071 (1995).